Phys 798C Spring 2022
Superconductivity
Relevant Review of Solid Sate Physics

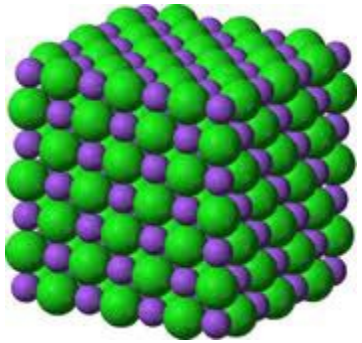
Free electron Fermi Gas model of a metal

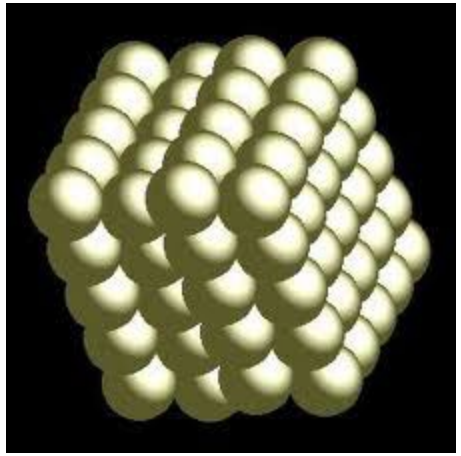
- **Single-particle states in a box**
- **Many identical Fermions – Pauli exclusion principle**
- **Density of States**
- **Fermi sea, Fermi surface**
- **Fermi energy, Fermi velocity, Fermi temperature**
- **Electronic heat capacity**

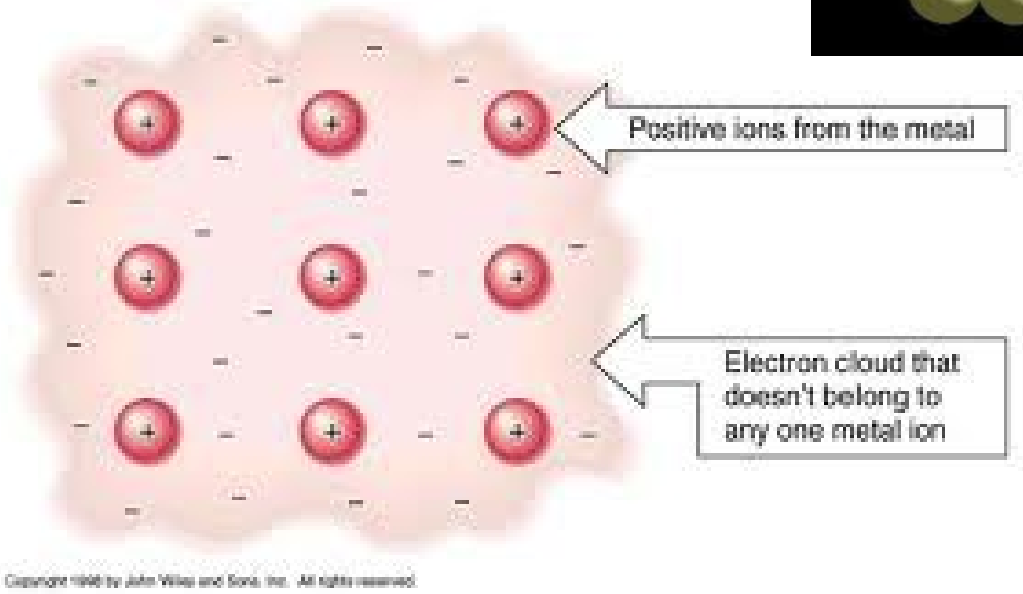
Phase transitions

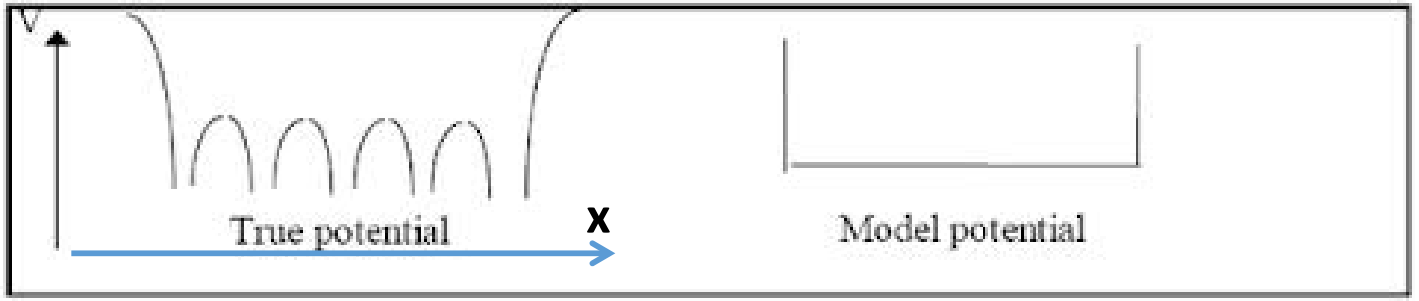
- **Order parameter, coherence length**
- **Spontaneous symmetry breaking**
- **Critical temperature**
- **Free energy expansion**
- **Fluctuations**
- **Critical Phenomena**

Free Electron Fermi Gas Model



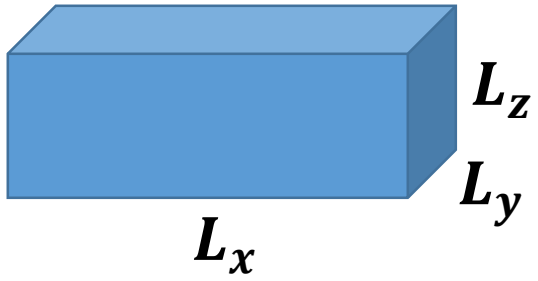






Free Electron Fermi Gas Model of a Metal

Particle in a box potential $V(x, y, z)$



$$\mathrm{H} = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} + V(x, y, z) \qquad \boldsymbol{H\psi = E\psi}$$

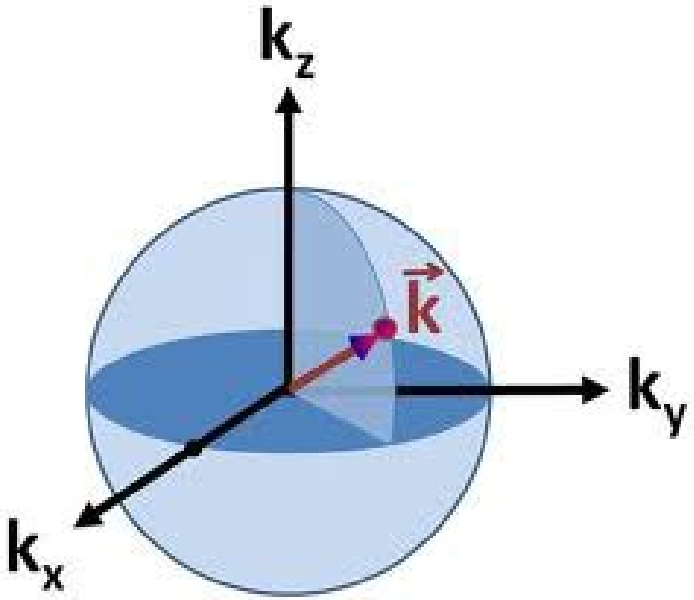
Running-wave solutions $\psi(x, y, z) \sim \frac{1}{\sqrt{V}}\, e^{ik_x x}\, e^{ik_y y}\, e^{ik_z z}$

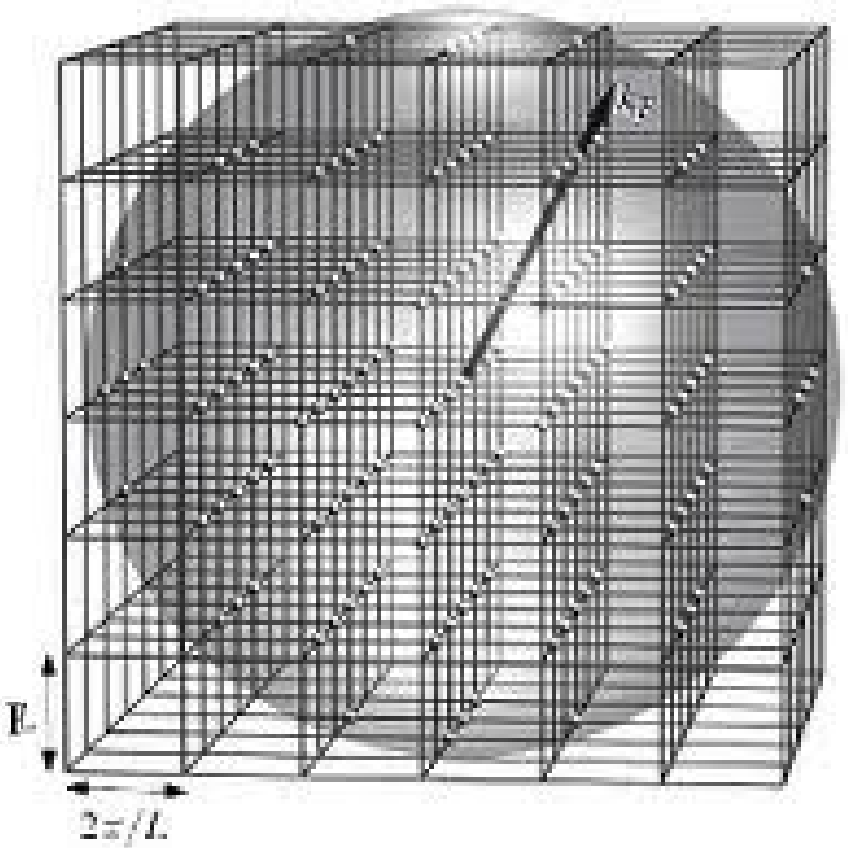
Born-von Karmen boundary conditions $\psi(x + L_x, y, z) = \psi(x, y, z)$ **etc.**

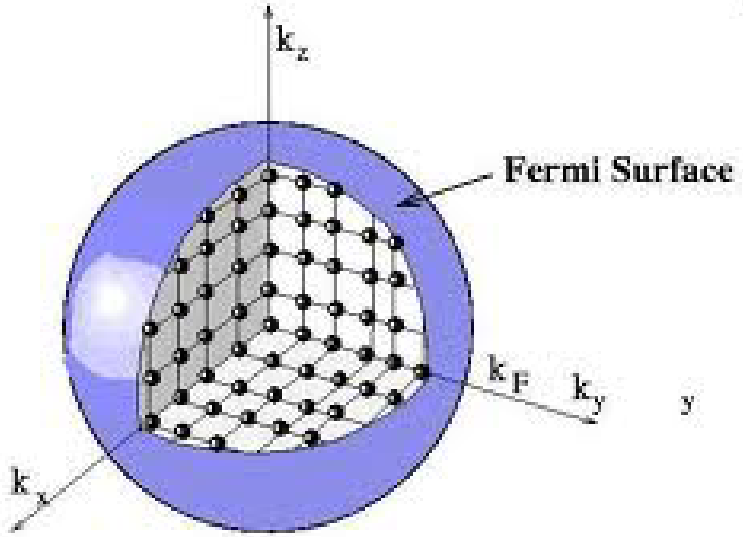
Solutions are also momentum eigenstates with $\vec{k} = 2\pi\left(\frac{n_x}{L_x}, \frac{n_y}{L_y}, \frac{n_z}{L_z}\right)$. $n_x = 0, \pm 1, \pm 2, \pm 3, \ldots$ **etc.**

Each single-particle state takes up a volume $\frac{2\pi}{L_x} \times \frac{2\pi}{L_y} \times \frac{2\pi}{L_z} = \frac{(2\pi)^3}{V}$ in k-space.

Single-particle states labeled by $\vec{k}, \sigma$ $\qquad \boldsymbol{E = \frac{\hbar^2 k^2}{2\, m}}$







k-space and the Fermi surface

Free Electron Fermi Gas Model of a Metal

continued

Now consider many (N~10^{23}) identical electrons occupying these single particle states

Pauli Exclusion Principle: No two particles in a multi-identical Fermion state with overlapping wavefunctions can have the same list of quantum numbers

Fill up all available states with electrons. The last occupied state is at the surface of the Fermi sea:

$$\frac{4}{3}\pi k_F^3 = \frac{Nq}{2}\frac{(2\pi)^3}{V} \qquad k_F = (3\pi^2\rho)^{1/3} \qquad \rho \equiv Nq/V \qquad q = \text{valence of each atom}$$

$$E_F = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2}{2m}(3\pi^2\rho)^{2/3} \qquad k_F \sim 10^8\ cm^{-1} \qquad v_F \sim 10^6\ m/s \qquad T_F = \frac{E_F}{k_B} \sim 10^5\ K \gg 300\ K$$

Metal	Valence (q)	E_F (eV)
Cu	1	7.0
Ag	1	5.5
Au	1	5.5
Li	1	4.7
Be	2	14.4
Al	3	11.66
Pb	4	9.4

Free Electron Fermi Gas Model of a Metal

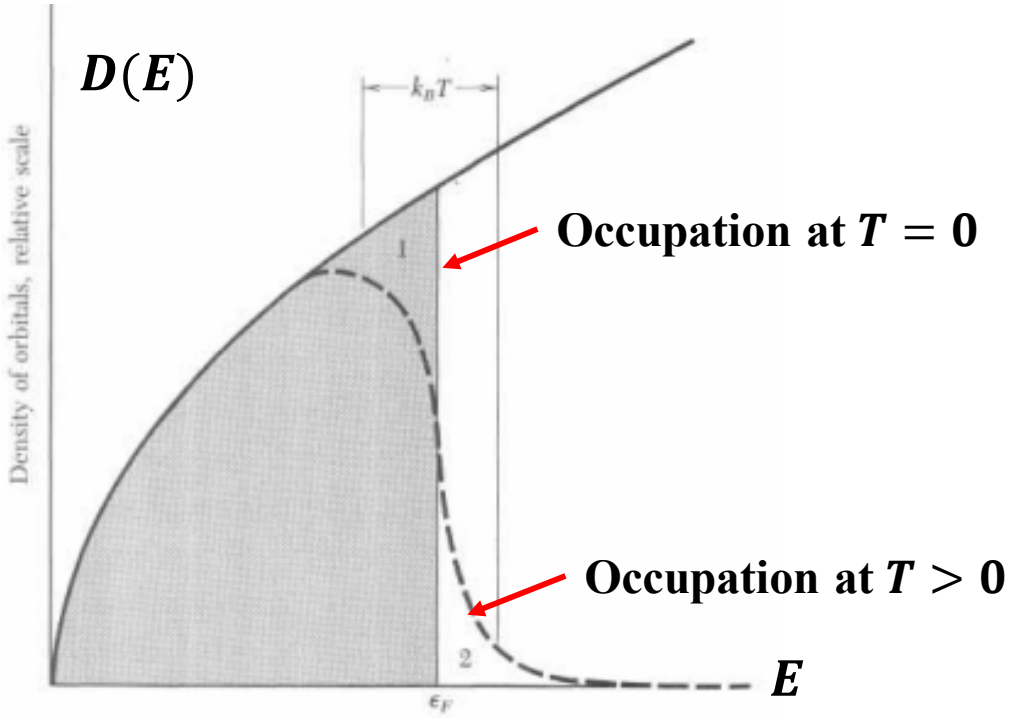
continued

Density of States $D(E)$

$$\sum_{n_x,n_y,n_z} \rightarrow \int D(E)dE$$

Number of States between E and $E + dE$:

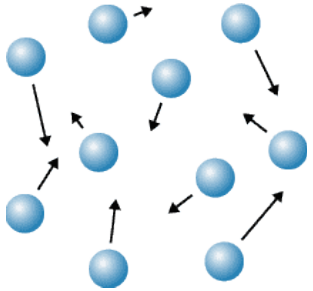
$$D(E)dE = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{E}\,dE$$



Kittel, Introduction to Solid State Physics, Chap. 6 of the 5th Edition

Free Electron Fermi Gas Model of a Metal

continued



Electronic Heat Capacity

Naïve free-electron gas model: By equipartition of energy $\frac{3}{2}k_BT$ of energy on average.

Total electronic energy: $U_{electrons} = N\frac{3}{2}k_BT$ **Electronic heat capacity:** $C_V = \left.\frac{dU}{dT}\right|_V = \frac{3}{2}k_BN$

Fermi/Pauli free-electron gas model:

$$U \sim (\#\text{ electrons in excited states}) \times (\text{energy acquired per electron}) = \left(N\frac{k_BT}{E_F}\right) \times (k_BT)$$

$$C_{V,electrons} = 2Nk_B\frac{k_BT}{E_F} \ll 2Nk_B$$

Free Electron Fermi Gas Model of a Metal

continued

Electronic Heat Capacity $C_{el} = \gamma T$

$$\gamma = \frac{1}{3}\pi^2 D(E_F) k_B^2$$

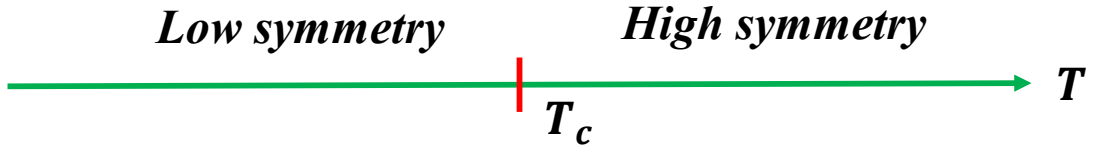
See Kittel, *Solid State Physics*, p. 165

For Aluminum $\gamma = 1.35 \ \frac{mJ}{mole\ K^2}$

Hence $D(E_F) = 0.57 \ \frac{states}{Al\ atom - eV}$

Phase Transitions

Second order phase transitions occur when a new state of reduced symmetry develops continuously from the disordered (high temperature) phase.



The ordered phase has a lower symmetry than the Hamiltonian—the phenomenon of *spontaneously broken symmetry*.

To describe the ordered state we introduce a *macroscopic order parameter* that describes the character and strength of the broken symmetry

Example: Ising ferromagnet: the Hamiltonian is invariant under all $s_i \rightarrow -s_i$, whereas the low temperature phase has a spontaneous magnetization, and so is not. A convenient order parameter is the total average spin $S = \sum_i \langle s_i \rangle$ or the magnetization $M = \mu S$.

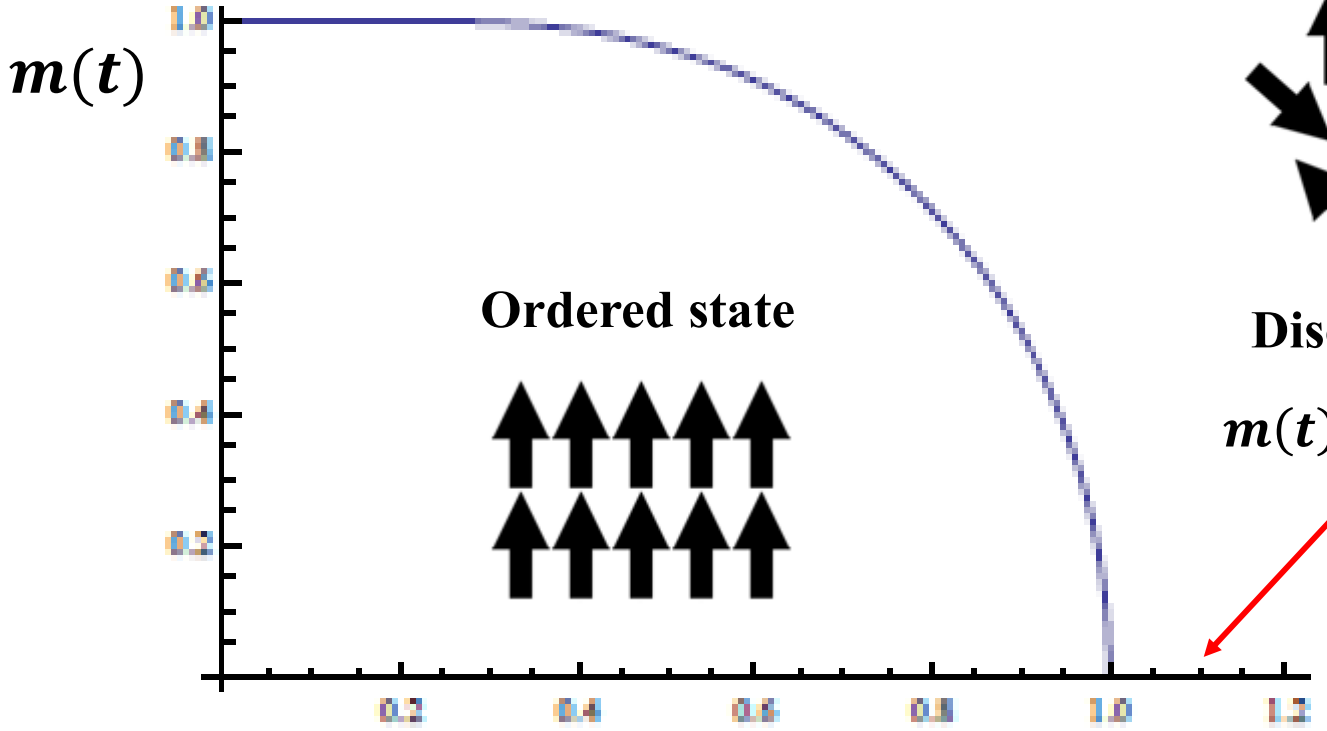
$$\mathcal{H} = J \sum_{i \neq j} \boldsymbol{s_i} \cdot \boldsymbol{s_j}$$

Example: Superfluid: the broken symmetry is the invariance of the (quantum) Hamiltonian under a phase change of the wave function. Since for a charge system a gauge transformation also changes the quantum phase, this is also known as broken gauge symmetry.

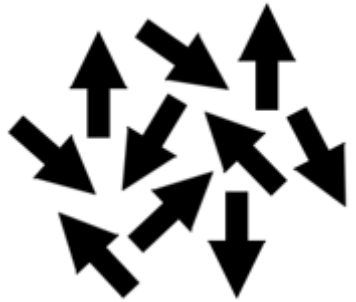
http://www.pmaweb.caltech.edu/~mcc/Ph127/b/Lecture6.pdf

Magnetization versus Temperature in the Mean Field Approximation

```
FindRoot[m == Tanh[m / 0.5], {m, 1}]

{m → 0.957504}

mag[t_] := m /. FindRoot[m == Tanh[m / t], {m, 1}]

mag[0.5]

0.957504

Plot[mag[t], {t, 0, 1.2}]
```



Ordered state

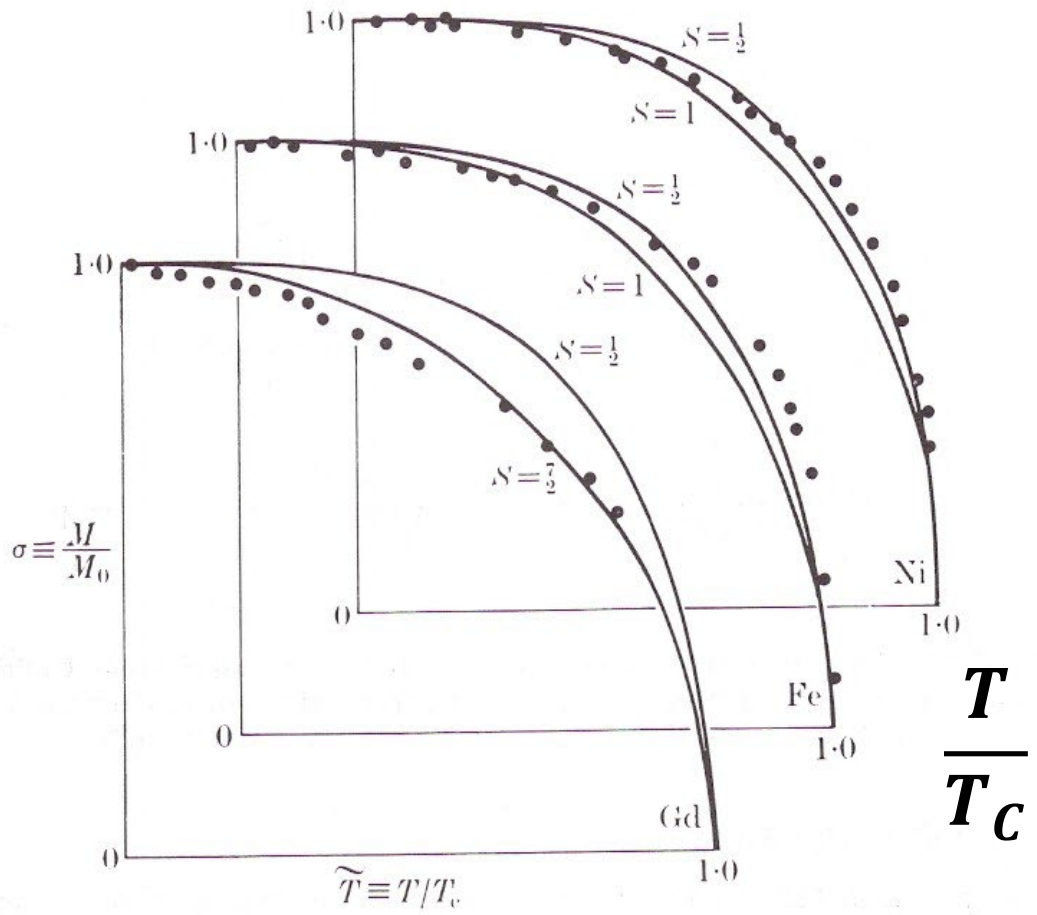


Disordered state

$m(t) = 0$ for $t > 1$

The Law of Corresponding States in Ferromagnetism

$\frac{M}{M_0}$

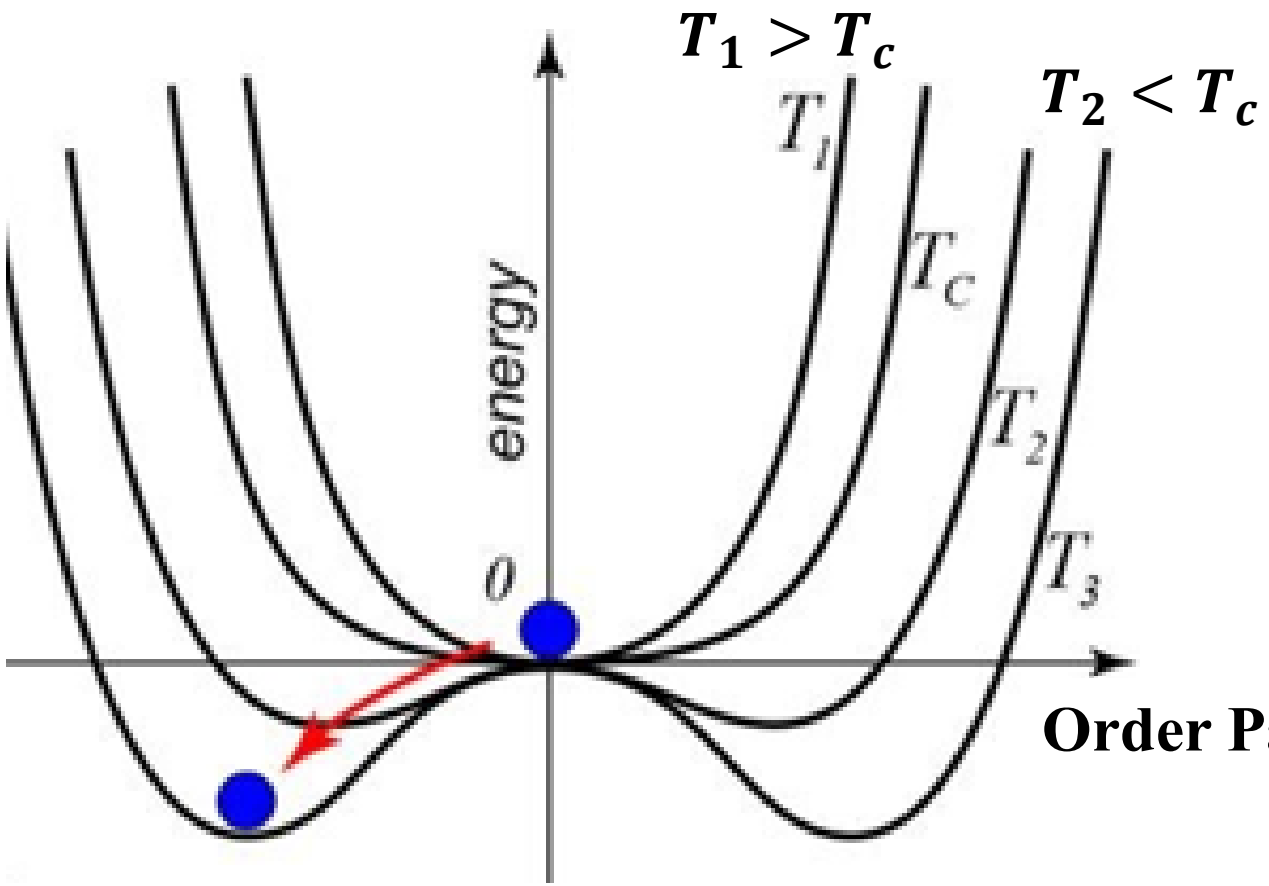


$\frac{T}{T_C}$

FIG. 6.5. Dependence of reduced magnetization $\sigma \equiv M/M_0 \equiv M(T, H)/M(0, 0)$ upon reduced temperature $\tilde{T} \equiv T/T_c$. The fact that there is a slightly different curve for each value of the spin quantum number S means that this law of corresponding states is valid only for a given value of S. The solid circles represent typical experimental data for Gd ($S \simeq \frac{7}{2}$), Fe ($S \simeq 1$), and Ni ($S \simeq \frac{1}{2}$). After Martin (1967).

H. E. Stanley, Intro. To Phase Transitions and Critical Phenomena, p. 88

Spontaneous Symmetry Breaking



We will use a free energy expansion in terms of a superconducting order parameter to describe the SC/Normal phase transition

https://doi.org/10.1091/mbc.e16-10-0739

Above T_C: Order Parameter = 0

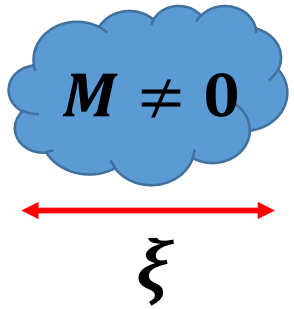
Below T_C: Order parameter, and it's value depends on details and fluctuations

Fluctuations and Critical Phenomena

Near a second order phase transitions it can cost little energy ($\sim k_B T_c$) to create a small and brief fluctuation of the order parameter

Consider $T > T_c$ where $M = 0$ in equilibrium

System borrows energy from the thermal bath to create a fluctuation:

$M = 0$ 

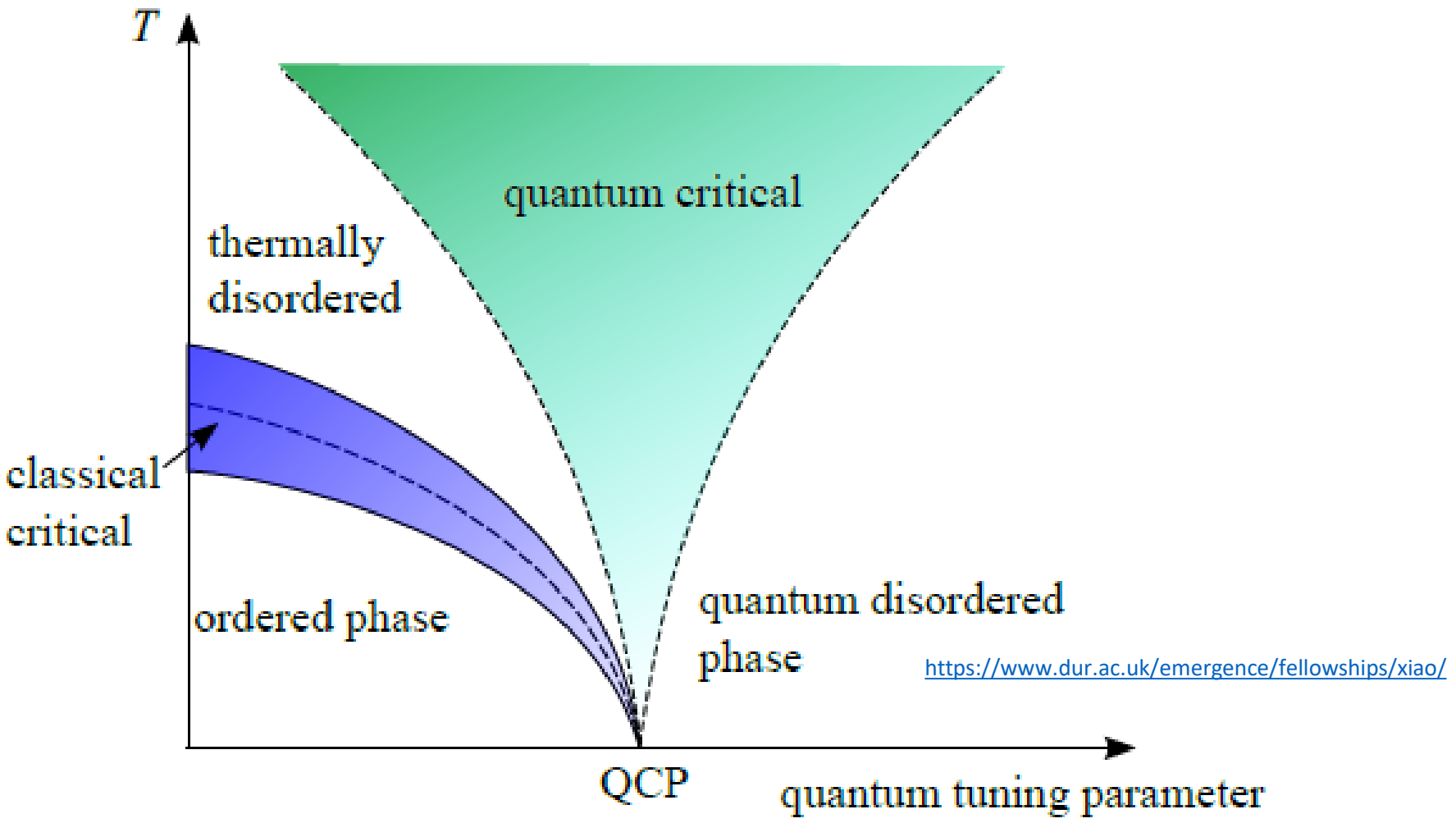
Lifetime: τ

These fluctuations change the physical properties of the material (susceptibility, conductivity, etc.)

Note that both ξ and τ diverge as $T \to T_c$

$$\xi \sim |T - T_C|^{-\nu}$$ **ν = static critical exponent**

$$\tau \sim |T - T_C|^{-\nu z}$$ **z = dynamic critical exponent**



https://www.dur.ac.uk/emergence/fellowships/xiao/